Stabilizing unstable discrete systems by a nonuniformly adaptive adjustment mechanism

Weihong Huang*
Nanyang Business School, Nanyang Technological University, Nanyang Avenue, Singapore 639798
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An adaptive adjustment mechanism with uniform adjustment speeds has been generalized to nonuniform speeds so that a broader class of discrete systems can be stabilized. The controllability issue in terms of partial adjustment is also explored.

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MOTIVATIONS

Controlling chaos, or more generally, stabilizing unstable dynamical systems, has always been an important topic in physics. Recent advances and developments can be seen from [1,2,3] and references therein. In [2], an adaptive adjustment mechanism (AAM) with uniform adjustment speed is proposed to stabilize an unstable multidimensional discrete system. This mechanism, while inherited from the adaptive expectation scheme widely applied in economics, possesses some unique advantages over the others in demanding neither *a priori* information about the system nor any external generated control signal and always forcing the system to converge to its generic periodic points. In this article, an AAM has been generalized with nonuniformly adjustment speeds. Such generalization ensures the stabilization of a more broader class of discrete systems. More importantly, the controllability issue in terms of partial adjustment is also explored.

Instead of focusing on the parameter ranges of some specific systems and discussing their controllability as most literature does, this paper will follow the spirit of [2] and continue our exploration directly to the nature of unstable periodic (fixed) points so as to address the controllability with respect to most fundamental internal structures. For the convenience of discussion, we start with some basic definitions and brief review of [2].

UNIFORMLY AAM

Consider an n-dimensional dynamical system defined by

$$\mathbf{X}_{t+1}=\mathbf{F}(\mathbf{X}_t), \tag{1}$$

where $\mathbf{X}_t=(x_{1t},x_{2t},\ldots,x_{nt})$, and $\mathbf{F}=(f_1,f_2,\ldots,f_n)$, with f_i being well-defined functions on a domain I^n.

Definition 1: By AAM we mean the following adjusted system:

$$\mathbf{X}_{t+1}=\tilde{\mathbf{F}}_\Gamma=(\mathbf{I}-\Gamma)\mathbf{F}(\mathbf{X_t})+\Gamma\mathbf{X_t}, \tag{2}$$

where $\Gamma=\text{diag}\{\gamma_1,\gamma_2,...,\gamma_n\}$ is a diagonal matrix and is referred to as an *adaptive parameter matrix* hereafter. The value of γ_i represents the adjustment speed for the ith variable ($i=1,2,...,n$) and is assumed to vary in the *conventional range* $[0, 1]$ and in the *generalized range* $[1, +\infty)$.

Re-expressing Eq. (2) as $\mathbf{X}_{t+1}=\mathbf{F}(\mathbf{X_t})+\Gamma[\mathbf{X_t}-\mathbf{F}(\mathbf{X_t})]$, we see that AAM is a type of linear feedback control so that adjustments are implemented whenever the relevant system variables wander away from their previous states.

Let $\overline{\mathbf{X}}$ be the fixed point of Eq. (1), that is, $\overline{\mathbf{X}}=\mathbf{F}(\overline{\mathbf{X}})$. It is easy to see that the system $\tilde{\mathbf{F}}_\Gamma(\mathbf{X}_t)$ shares exactly the same set of fixed points of $\mathbf{F}$, that is, $\overline{\mathbf{X}}=\tilde{\mathbf{F}}_\Gamma(\overline{\mathbf{X}})$, which will be referred to as *the generic property* for later reference.

Denote $\mathcal{J}(\overline{\mathbf{X}})$ as the Jacobian matrix of the original system $\mathbf{F}$ evaluated at $\overline{\mathbf{X}}$ with $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ as the n roots of the characteristic equation, i.e.,

$$|\lambda\mathbf{I}-\mathcal{J}(\tilde{\mathbf{X}})|=\prod_{j=1}^{n}(\lambda-\lambda_j)=0,$$

where $\mathbf{I}$ is a unit matrix.

The stability of a fixed point $\overline{\mathbf{X}}$ is jointly determined by all the eigenvalues $\{\lambda_j\}$. Let $|\lambda_{\max}|=\max_j|\lambda_j|$. Mathematically, the fixed point $\overline{\mathbf{X}}$ is stable if $|\lambda_{\max}|<1$.

We are only concerned with the unstable fixed points, that is, the fixed points with $|\lambda_{\max}|\geqslant 1$. Denote a pair of complex conjugates λ_j and $\overline{\lambda_j}$ by

$$\lambda_j=a_j+b_j\mathbf{i},\quad \overline{\lambda_j}=a_j-b_j\mathbf{i},$$

with the modules $|\lambda_j|=|\overline{\lambda_j}|=\sqrt{a_j^2+b_j^2}$.

An unstable fixed point can be classified according to the modulus of related eigenvalues:

Definition 2 (classification of unstable fixed points): *type-I unstable fixed points*, $a_j<1$, for all j, i.e., the fixed points with all eigenvalues less than unity in real parts; *type-II unstable fixed points*, $a_j>1$, for all j, i.e., the fixed points with all eigenvalues greater than unity in real parts; *type-III unstable fixed points*, $a_i>1$, $a_j<1$ for some i, j, i.e., the fixed points with some real parts greater than unity, others less than unity in real parts; *type-IV unstable fixed points*, there exists at least one j such that either $a_j=1$ or $\lambda_j=1$, i.e., the fixed points with unity eigenvalues.

Let $\tilde{J}(\overline{\mathbf{X}})$ be the Jacobian matrix of the process $\tilde{\mathbf{F}}$ evaluated at $\overline{\mathbf{X}}$ and $\{\tilde{\lambda}_1,\tilde{\lambda}_2,\ldots,\tilde{\lambda}_n\}$ be the related eigenvalues, so that

*Electronic address: awhhuang@ntu.edu.sg

$$|\lambda \mathbf{I}-\tilde{\mathcal{J}}(\overline{\mathbf{X}})|=\prod_{j=1}^{n}(\lambda-\tilde{\lambda}_j)=0. \qquad (3)$$

The objective of AAM is to stabilize an unstable fixed point such that, after introducing an appropriate adaptive parameter matrix $\Gamma=\mathrm{diag}\{\gamma_1,\gamma_2,...,\gamma_n\}$, the new eigenvalues $\tilde{\lambda}_j$, $j=1,2,...,n$ become less than unity in modulus.

For an one-dimensional discrete system and its adjusted counterpart, at a fixed point $\overline{x}$, there exists a simple one-to-one relationship between $\lambda=f'(\overline{x})$ and $\tilde{\lambda}=\tilde{f}'(\overline{x})$:

$$\tilde{\lambda}=(1-\lambda)+\gamma. \qquad (4)$$

For a multidimensional systems, the one-to-one relationship between λ_j, γ_j, and $\tilde{\lambda}_j$ like identity (4) does not exists in general, which makes the analysis of the effect of each γ_j on $\tilde{\lambda}_j$ turn out to be extremely difficult. One exception lies in the case of the uniformly AAM, that is,

$$\Gamma=\mathrm{diag}\ \{\gamma,\gamma,\dots,\gamma\}=\gamma\mathbf{I}_n,$$

that is, all variables are adjusted with the same speed:

$$X_{t+1}=\tilde{F}=(1-\gamma)F(X_t)=\gamma X_{\mathbf{t}}. \qquad (5)$$

The following corollary is shown in [2].

Corollary 1. For each and every fixed point of F and $\tilde{F}$, there exists the following one-to-one correspondence between their eigenvalues:

$$\tilde{\lambda}_j=(1-\gamma)\lambda_j+\gamma, \quad j=1,2,...,n. \qquad (6)$$

Theorem 1 and the generic property together enable us to adjust the eigenvalues to become less than unity in modulus by suitable choice of a single adaptive parameter γ only.

Corollary 2. For a n-dimensional dynamical system $X_{t+1}=F(X_t)$, an unstable fixed point $\overline{X}$ can be stabilized through uniformly adaptive adjustment defined by Eq. (5) if and only if $\overline{X}$ is either a type-I fixed point ($a_j<1$ for all $j=1,2,...,n$) or a type-II fixed point ($a_j>1$ for all $j=1,2,...,n$)

Example 1. Type-I fixed point stabilized by uniformly AAM.

Consider the following n-dimensional discrete systems:

$$x_{it+1}=a_i-\tfrac{1}{2}\sum_{j\neq i}^{n}x_{jt}, \quad \text{for } i=1,2,...,n.$$

Due to the linearity, the Jacobian matrix is a constant matrix given by

$$J=\begin{pmatrix} 0 & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{2} & 0 & -\frac{1}{2}\ \cdots & -\frac{1}{2} \\ \vdots & \vdots & \cdots & \vdots \\ -\frac{1}{2} & -\frac{1}{2} & \cdots & 0 \end{pmatrix}$$

It can be verified that the eigenvalue of J is $\lambda_1=-(n-1)/2$ and $\lambda_i=\frac{1}{2}$, for $i=2,3,...,n$. Therefore, the system is unstable when $n\geqslant 3$. Since all eigenvalues are less than unity (type-I fixed point), the system can be stabilized through uniformly adaptive adjustment. Actually, the critical parameter $\overline{\gamma}$ is equal to $\overline{\gamma}=(\lambda_1+1)/(\lambda_1-1)=(n-3)/(n+1)$, that is, the system can converge to a fixed point if $\gamma\in((n-3)/(n+1),1)$.

NONUNIFORMLY AAM

Corollary 2 provides us with invaluable insight about the effectiveness of nonuniformly AAM given by Eq. (2).

If we assume that f_j is continuous for almost all points, for $j=1,2,...,n$, then it is easy to see that, if all γ_j, $j=1,2,...,n$, are sufficient close to γ^*, the dominant eigenvalue of $\tilde{J}$ can still be guaranteed to be less than unity in modulus. Consequently, we have the following.

Theorem 1. For a n-dimensional dynamical system $X_{t+1}=F(X_t)$, if an unstable fixed point $\overline{X}$ is either a type-I fixed point ($a_j<1$ for all $j=1,2,...,n$) or a type-II fixed point ($a_j>1$ for all $j=1,2,...,n$), it can always be stabilized through a nonuniformly AAM with suitable choice of adaptive parameter matrix.

Notice that type-IV fixed points are essentially related to bifurcation phenomenon, which can be easily overcome (to be changed into either a type-I or a type-II fixed point) through varying the original system's parameters; we shall not discuss this in detail. So it does not deserve special discussion. We shall thus concentrate on the type-III fixed point, which turns out to be uncontrollable by uniformly AAM.

By intuition, it seems to be possible to stabilize any type of fixed point by a suitable adaptive parameter matrix Γ, with some γ_j's in the conventional range, others in the generalized range. Formally, it is questioned that, for a given nonlinear system (1), if its fixed points are of type III or type IV, whether there always exists an adaptive parameter matrix $\Gamma=\mathrm{diag}\{\gamma_1,\gamma_2,...,\gamma_n\}$, with at least one i and j such that $\gamma_i\neq\gamma_j$, such that the adjusted system (2) is stabilized at the same fixed point. The answer is unfortunately negative.

Mathematically, however, a simple relationship between the original eigenvalues and new eigenvalues analogous to identity (6) can be obtained only for some special situations such as recursive systems (to be discussed in the following). Now that all variables are dependent on each other, on one hand, stability may be easily achieved by adaptively adjusting only parts of the variables. On the other hand, if each variable is adjusted independently without necessary coordination, the overall effects become totally unpredictably erratic.

To exemplify these remarks, we start with the examination of a two-dimensional discrete system.

Let $J(\overline{\mathbf{X}})$ be the Jacobian matrix associated with a fixed point $\overline{\mathbf{X}}$ of some two-dimensional system:

$$\mathcal{J}(\overline{X})=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Denote

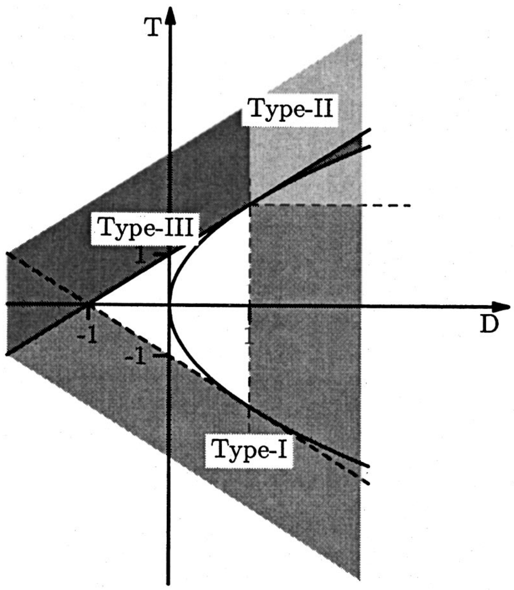

FIG. 1. Distribution of fixed points.

$$\mathcal{T}=a+d=\text{trace of } \mathcal{J},$$

$$\mathcal{D}=ad-bc=\text{determinant of } \mathcal{J},$$

$$\mathcal{H}=\mathcal{T}^2-4D,$$

then eigenvalues of $J(\overline{\mathbf{X}})$ can be expressed in terms of these invariants, as follows:

$$\lambda_{1,2}=\tfrac{1}{2}\,(\mathcal{T}\pm\sqrt{\mathcal{H}})=\tfrac{1}{2}\,(\mathcal{T}\pm\sqrt{\mathcal{T}^2-4\mathcal{D}}). \tag{7}$$

The stability regime and distribution of unstable fixed points can be depicted in a $(\mathcal{T},\mathcal{D})$ plane, which is sketched in Fig. 1.

It is shown that a type-IV fixed point is represented by the divergence bifurcation boundary $\mathcal{T}-\mathcal{D}=1$, while a type-III fixed point $(\lambda_1>1,\ \lambda_2<1)$ occurs only under two situations: (i) $\mathcal{D}<1$ and $\mathcal{T}-\mathcal{D}>1$; and (ii) $\mathcal{D}>1$, $\mathcal{T}-\mathcal{D}<1$, but $\mathcal{H}>0$.

With adaptive adjustment of $\Gamma=\text{diag}\{\gamma_1,\gamma_2\}$, the Jacobian matrix becomes

$$\tilde{J}(\overline{\mathbf{X}})=\begin{pmatrix}(1-\gamma_1)a+\gamma_1 & (1-\gamma_1)b\\ (1-\gamma_2)c & (1-\gamma_2)d+\gamma_2\end{pmatrix}, \tag{8}$$

which gives the eigenvalues $\overline{\lambda_{1,2}}$ pair as $\overline{\lambda_{1,2}}=1/2(\tilde{\mathcal{T}}\pm\sqrt{\tilde{\mathcal{T}}^2-4\tilde{\mathcal{D}}})$ where

$$\tilde{\mathcal{T}}=\mathcal{T}+\gamma_1(1-a)+\gamma_2(1-d),$$

and

$$\tilde{\mathcal{D}}=(1-\gamma_1)(1-\gamma_2)\mathcal{D}+(1-\mathcal{T})\gamma_1\gamma_2+a\gamma_2+\gamma_1 d.$$

We see that, even for a two-dimensional system, the relationship between adjustment parameters $\gamma_{1,2}$, original eigenvalues $\lambda_{1,2}$, and new eigenvalues $\tilde{\lambda}_{1,2}$ become very complicated.

It can be verified that the simple relationship

$$\tilde{\lambda}_j=(1-\gamma_j)\lambda_j+\gamma_j,\quad j=1,2$$

exists if and only if one of the following situations occurs: (i) $\gamma_1=\gamma_2$, i.e., *uniformly adjustment*; and (ii) $bc=0$, i.e., *recursive systems*.

In general, when $\gamma_1\neq\gamma_2$, each eigenvalue is affected by both adjustment parameters symmetrically; the interaction of these parameters makes the sensitivity analysis of overall effects become quite difficult.

Now that uniformly adaptive adjustment, which is a special case of nonuniformly AAM, can stabilize both the type-I and type-II fixed points, by the continuity argument, we can assure the existence of a nonuniform adjustment parameter matrix $\Gamma=\text{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ [with at least one pair of (i,j) such that $\gamma_i\neq\gamma_j$] that can stabilize type-I and type-II fixed points.

Then what remains unsolved are type-III fixed points, that is, the fixed points with part of the eigenvalues are greater than or equal to unity but the rest are less than unity. Several issues need to be resolved.

At first, even though we have shown that a type-III fixed point cannot be stabilized through uniformly adaptive adjustment, we are still not sure whether they can be stabilized through a combination of adaptive parameters that are not identical but all in the same range (either in conventional range or generalized range). An ''impossibility'' is shown for a two-dimensional system.

Actually, for the Jacobian matrices (7) and (8), we have

$$\tilde{\mathcal{T}}-\tilde{\mathcal{D}}=(1-\gamma_1)(1-\gamma_2)(\mathcal{T}-\mathcal{D}-1)+1,$$

which implies the following.

(1) A type-IV fixed point cannot be stabilized by any (γ_1,γ_2), owing to the fact that $\tilde{\mathcal{T}}-\tilde{\mathcal{D}}=1$ if $\mathcal{T}-\mathcal{D}=1$.

(2) A type-III fixed point with $\mathcal{D}<1$ and $\mathcal{T}-\mathcal{D}>1$ can only be stabilized through a combination of (γ_1,γ_2) satisfying the inequality $(1-\gamma_1)(1-\gamma_2)<0$, that is, one takes values in the conventional range, the other takes values in the generalized range.

Second, *does there exist any special form systems for which AAM always works—not only for type-I or type-II fixed points, but also for type-III fixed points*? The answer is definitely ''Yes.'' One such system is the recursive system that has been widely applied in social science.

A nonlinear system $\mathbf{F}(\mathbf{X})=\{f_1(\mathbf{X}),f_2(\mathbf{X}),\ldots,f_n(\mathbf{X})\}$, with $\mathbf{X}=(x_1,x_2,\ldots,x_n)$, is *recursive* if f_i depends only on the first i variables, that is,

$$\begin{aligned}x_{1t+1}&=f_1(x_{1t}),\\ x_{2t+1}&=f_2(x_{1t},x_{2t}),\\ &\vdots\\ x_{kt+1}&=f_k(x_{1t},x_{2t},\ldots,x_{kt}),\\ &\vdots\\ x_{nt+1}&=f_n(x_{1t},x_{2t},\ldots,x_{nt}).\end{aligned} \tag{9}$$

Theorem 2 (recursive systems). For an n-dimensional recursive system defined by Eq. (9), if $df_i/dx_i|_{X=\overline{X}}\neq 1$, for i

$=1,2,...,n$, then there always exists an adaptive parameter matrix $\Gamma=\text{diag}\,\{\gamma_1,\gamma_2,...,\gamma_n\}$ such that the adjusted system

$$\mathbf{X}_{t+1}=\overline{\mathbf{F}}_\Gamma=(\mathbf{I}-\Gamma)\mathbf{F}(\mathbf{X_t})+\Gamma\mathbf{X_t} \tag{10}$$

can be stabilized to its generic fixed point $\overline{\mathbf{X}}$.

Proof. If $\mathbf{F}$ is recursive, then at the fixed point $\overline{X}$, its Jacobian matrix is an upper or lower triangular matrix. Following the definition of Eq. (10),

$$\mathcal{J}(\overline{\mathbf{X}})=\begin{pmatrix} \frac{df_1}{dx_1} & 0 & \cdots & 0 \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_n} \end{pmatrix}_{\mathbf{X}=\overline{\mathbf{X}}}, \tag{11}$$

with eigenvalues $\lambda_i=df_i/dx_i$, $i=1,2,...,n$. At the same fixed point, the Jacobian matrix for the adjusted system (10) becomes

$$\widetilde{\mathcal{J}}(\overline{\mathbf{X}})=\begin{pmatrix} (1-\gamma_1)\frac{df_1}{dx_1}+\gamma_1 & 0 & \cdots & 0 \\ (1-\gamma_2)\frac{df_2}{dx_1} & (1-\gamma_2)\frac{df_2}{dx_2}+\gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (1-\gamma)\frac{df_n}{dx_1} & (1-\gamma_n)\frac{df_n}{dx_2} & \cdots & (1-\gamma_n)\frac{df_n}{dx_n}+\gamma_n \end{pmatrix}_{X=\overline{X}}, \tag{12}$$

which gives rise to the eigenvalues

$$\tilde{\lambda}_i=(1-\gamma)\frac{df_i}{dx_i}\bigg|_{X=\overline{X}}+\gamma_i, \quad i=1,2,...,n.$$

It follows from the discussion in previous sections, if $df_i/dx_i|_{X=\overline{X}}\neq 1$, that is, the fixed point is not of type IV, there always exists a $\gamma_i>0$ such that $|\tilde{\lambda}_i|<1$, for all $i=1,2,...,n$. Q.E.D.

Theorem 2 serves both as an example that a type-III fixed point can be stabilized through nonuniformly adaptive adjustment and as an example that a type-IV fixed point that cannot be stabilized through AAM.

CONTROLLABILITY OF AAM

The last issue deserving special attention is *controllability*. As we have commented before, if a multidimensional system is not symmetrical, the effect of each adaptive parameter γ_i, $i=1,2,...,n$ on the stability will be different. There exist situations in which some of adaptive parameters are *dispensable*, that is, the system can still be stabilized if these variables are not adjusted ($\gamma_i=0$). On the other hand, there are some critical adjustment parameters that are *indispensable*, that is, the stability cannot be achieved if any one of them equals to zero. This point can be clearly illustrated through the following three examples.

Example 2. Part of adjustment parameters are indispensable.

For a two-dimensional system, if its Jacobian at a fixed point $\overline{\mathbf{X}}$ is given by

$$\mathcal{J}(\overline{\mathbf{X}})=\begin{pmatrix} \lambda_1+\lambda_2 & -\lambda_1\lambda_2 \\ 1 & 0 \end{pmatrix},$$

then its counterpart from adaptive adjustment will produce $\tilde{\lambda}_{1,2}=\frac{1}{2}(\widetilde{\mathcal{T}}\pm\sqrt{\widetilde{\mathcal{T}}^2-4\widetilde{\mathcal{D}}})$, where

$$\widetilde{\mathcal{T}}=\gamma_1+(1-\gamma_1)(\lambda_1+\lambda_2),$$

$$\widetilde{\mathcal{D}}=(1-\gamma_1)(1-\gamma_2)\lambda_1\lambda_2.$$

In this case, the adjustment parameter γ_2 has no effect on the real part, so there exist some cases in which the dynamics cannot be controlled by γ_2 alone.

In fact, if $\gamma_1=0$, we have $\tilde{\lambda}_{1,2}=1/2[\lambda_1+\lambda_2\pm\sqrt{(\lambda_1+\lambda_2)^2+4\lambda_1\lambda_2\gamma_2}]$. Either $|\lambda_1+\lambda_2|>2$, or $\lambda_1\lambda_2>0$, γ_2 will become ineffective. Therefore, γ_1 is indispensable.

For example, the Henon process $\mathbf{X}_{t+1}=\theta(\mathbf{X}_t)$, defined by

$$x_{1t+1}=\frac{7}{5}+\frac{3}{10}x_{2t}-x_{1t}^2,$$

$$x_{2l+1}=x_{1t}, \tag{13}$$

has two fixed points: $\overline{\mathbf{X}}_1\approx(0.8839,0.8839)$ with eigenvalues $\{\lambda_1^{(1)},\lambda_2^{(1)}\}=\{0.156,-1.924\}$, and $\overline{\mathbf{X}}_2\approx(-1.5839,-1.5839)$ with eigenvalues $\{\lambda_1^{(2)},\lambda_2^{(2)}\}=\{3.26,-0.92\}$, respectively.

The fixed point $\overline{\mathbf{X}}_1$ is of type I, so it can be stabilized through uniformly AAM (referring to [2] for computer simulations).

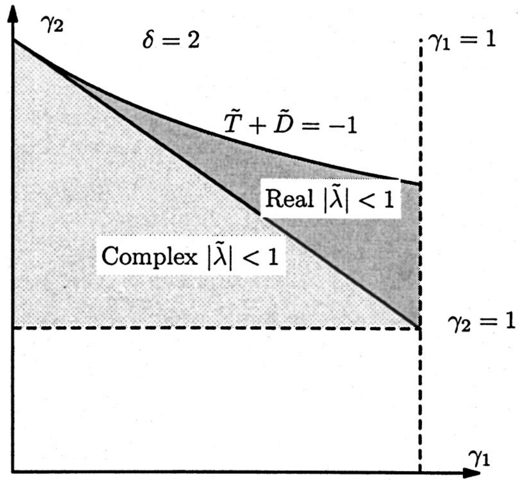

FIG. 2. Asymmetric stabilization.

But $\overline{\mathbf{X}}_2$ is a type-III fixed point, so it can only be stabilized through nonuniformly AAM,

$$x_{1t+1} = (1-\gamma_1)\left[\frac{7}{5} + \frac{3}{10}x_{2t} - x_{1t}^2\right] + \gamma_1 x_{1t},$$

$$x_{2t+1} = (1-\gamma_2)x_{1l} + \gamma x_{2t}. \tag{14}$$

The fact that $|\lambda_1^{(2)} + \lambda_2^{(2)}| > 2$ implies that γ_1 is indispensable.

Example 3. Both adjustment parameters are indispensable.

If the Jacobian at the fixed point $\overline{\mathbf{X}}$ is given by

$$\mathcal{J}(\overline{\mathbf{X}}) = \begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix},$$

the eigenvalues will be a positive reciprocal pair due to the facts that

$$\lambda_1 \lambda_2 = \mathcal{D} = 1,$$

and

$$\lambda_1 + \lambda_2 = \mathcal{T} = 2 + \alpha\beta,$$

which suggests that the fixed point (0, 0) is a type-III fixed point.

The Jacobian matrix from adaptive adjustment is

$$\tilde{\mathcal{J}} = \begin{pmatrix} 1 & (1-\gamma_1)\alpha \\ (1-\gamma_2)\beta & 1+(1-\gamma_2)\alpha\beta \end{pmatrix}.$$

If we let $\delta = \alpha\beta$, we have $\tilde{\mathcal{D}} = |\tilde{\mathcal{J}}| = 1 + \delta\gamma_1(1-\gamma_2)$. Therefore, $\tilde{\mathcal{D}} > 1$ holds in the case of $\gamma_1\gamma_2 = 0$, which implies that neither γ_1 nor γ_2 alone has enough power to force the system to converge to the fixed point. The stability can only be achieved only when γ_1 is in the conventional range, while γ_2 stays in the generalized range.

Stabilization regime is jointly given by *flip bifurcation* boundary ($\tilde{\lambda}_2 = -1$): $\tilde{\mathcal{T}} + \tilde{\mathcal{D}} = -1$, and $\gamma_2 > 1$, where $\tilde{\mathcal{T}} = 2 + (1-\gamma_2)\delta$. A typical example is illustrated in Fig. 2 for $\delta = 2$.

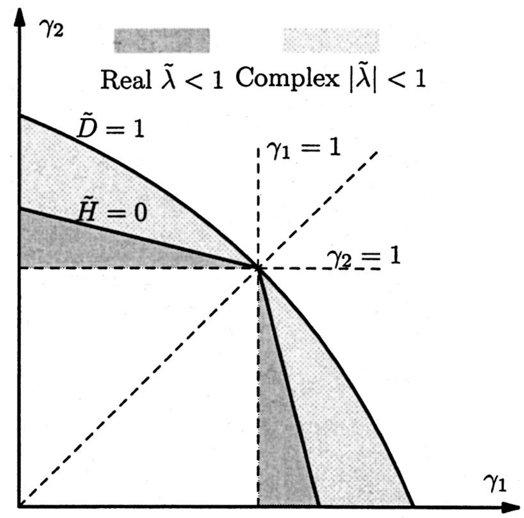

FIG. 3. Symmetrical stabilization.

Example 4. None of the adjustment parameters are indispensable.

Consider the case that

$$\mathcal{J}(\overline{\mathbf{X}}) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix},$$

with $\lambda > 1$, then the eigenvalues $\lambda_{1,2} = \pm\lambda$, that is, the fixed point $\overline{\mathbf{X}}$ is of type III. By nonuniformly AAM, the eigenvalue pair changes to

$$\tilde{\lambda}_{1,2} = \frac{\gamma_1 + \gamma_2}{2} \pm \tfrac{1}{2}\sqrt{(\gamma_1 - \gamma_2)^2 + 4(1-\gamma_1)(1-\gamma_2)\lambda^2}.$$

Therefore, both adjustment parameters have equal power in stabilizing the dynamics, which suggests that the stabilization regime in (γ_1, γ_2) plane is symmetrical. A typical stabilization regime is illustrated in Fig. 3. Notice that either parameter alone can stabilize the system by taking an adaptive speed in the generalized range.

FUTURE STUDY

Compared to other algorithms so far proposed in the literature, the adaptive adjustment mechanism possesses some unique advantages. First, it requires neither *a priori* information about the system nor any external generated control signal. Second, it is easy to implement in practice. Last but not least, it forces the system to converge to its generic fixed points.

As we can see from the examples of the last section, when nonuniformly adjustment is required for stabilizing a type-III fixed point, the adjustment parameters should distribute in conventional range and generalized range with certain ratio. If the original system is dynamically bounded in the sense that its trajectory is constrained to a certain subspace, which occurs with chaotic systems, gradually increasing an adaptive parameter from zero onwards will not destroy such ''boundedness.'' But if some of adjustment parameters inappropriately exceed unity, bounded dynamics may not be guaranteed. It would be the future research to design an appropriate learning mechanism or coordination mechanism so that stabilization can be accomplished without risk of destroying the original system. The necessary and/or sufficient conditions for a type-III fixed point to be stabilized also deserve further study.

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